Factorization of a linear partial differential equation into 1st order and quadratic forms

T.Srinivasarao
Assistant Professor
Department of Mathematics
University College of Science & Technology
Adikavi Nannaya University
India - 533296

K.Jaya rama Krishna
Associate Professor in Mathematics
HOD of Basic Science &Humanities Department
Wise Engineering College

Abstract:

Recollect the fundamental theorem of Algebra that gives n linear factors of an n^{th} degree polynomial in the complex plane if not in the real plane. But the complex conjugates are allowed in the canonical factorization of a 2nd order linear partial differential equation. Therefore, an n^{th} order linear partial differential equation can be seen as n linear 1st order partial differential equations and the reduced quadratic canonical forms with constant or variable coefficients in the n – dimensional real space. The variable coefficients are the functions of independent variables. The elliptical canonical form of a 2nd order linear partial differential equation is obtained from the complex roots, it can better be viewed that the n^{th} order linear partial differential equation can be factorized completely in an n – dimensional complex space in view of fundamental theorem of algebra. If $\omega(D, D_1)$ is a non constant polynomial, then there is a pair of numbers a and b satisfy $\omega(a, b) = 0$

1. ***Introduction:*** An n^{th} order ordinary linear differential equation can be written as the product of n, 1st order linear differential equations. With the same spirit, a linear partial differential equation of order n with constant or variable coefficients can be written as the product of linear 1st order differential equations of any multiplicity and irreducible 2nd order linear partial differential equations. But, their respective reduced canonical forms are the quadratic forms that the solution surface is a suitable type of conic. These quadratic forms can once again be categorized with respect to the discrimiant that decides the conical surfaces of the solutions of the partial differential equation which are the intersections of the solution surfaces of the linear and quadratic factors of the n^{th} order partial differential equation as shown in 3.5.

Dividing the given linear partial differential equation $\omega(D, D_1, E, F, G)u = f(\theta, \vartheta)$ of order n with τ_{rs} as a function of θ and ϑ for each r and s,

$D = \frac{\partial}{\partial\theta}, D_1 = \frac{\partial}{\partial\vartheta}, E = \frac{\partial^2}{\partial\theta^2}, F = \frac{\partial^2}{\partial\theta\partial\vartheta}$ and $G = \frac{\partial^2}{\partial\vartheta^2}$ …… 1.1

2. ***2nd order linear factors***: See that each quadratic form can be compared to $RE + SF + TG = f(\theta, \vartheta, u, D, D_1)$ and depending on the discriminant the roots of the quadratic form will determine the solution surfaces that are hyperbolic, parabolic and elliptic from the canonical form that has a new set of variables
$\rho(\theta,\vartheta) = \rho, \varphi(\theta,\vartheta) = \varphi, \mu(\rho,\varphi) = \mu$ …… 2.1

$E + 2F + G = 0$ has the discriminant $S^2 - 4RT = 0$. So, the quadratic form has the roots
$\frac{\rho_\theta}{\rho_\vartheta} = \tau_1 = \frac{\varphi_\theta}{\varphi_\vartheta}$ leading the solution surface to be a paraboloid opening up. …… 2.2

$E - \theta^2 G = 0$has the discriminant $S^2 - 4RT > 0$. The roots of the quadratic form has the roots
$\frac{\rho_\theta}{\rho_\vartheta} = \tau_1, \frac{\varphi_\theta}{\varphi_\vartheta} = \tau_2$ leading the solution surface to be a hyperboloid …… 2.3

$E + \theta^2 G$ has the discriminant $S^2 - 4RT < 0$
The roots $\frac{\rho_\theta}{\rho_\vartheta} = \tau_1, \frac{\varphi_\theta}{\varphi_\vartheta} = \tau_2$ are the complex conjugates that allow the elliptic canonical form leading the solution surface to be a ellipsoid …… 2.4

The convex region that is enclosed by the solution surfaces will be the feasible region of the character function $u(\theta,\vartheta)$ in the *n* – dimensional complex space.

3. ***Application of Fundamental Theorem of Algebra to nth order linear partial differential equation:***

Proposition.3.1.: An *n*th order linear partial differential equation can be factorized as 1st order and quadratic forms whose solutions form a convex region enclosed by conic surfaces.

Lemma 3.2.: The boundary conditions of linear partial differential equation or a quadratic form can be obtained from the extreme value |Ø| where $\pi(D, D_1, E, F, G)u = \emptyset$

Proposition.3.3.: A convex region in an *n* – dimensional space is enclosed by either conic surfaces or planar surfaces that are the solutions of 2nd order and 1st order partial differential equations respectively whose sum of orders is equal to *n*.

Corollary.3.4.: if $P(D, D_1, E, F, G) = \prod_{r,s=1}^{q} c_r(\alpha_r D + \beta_r D_1 + \gamma_r)^{k_r}(\alpha_s E + \beta_s F + \gamma_s G)^{m_s}$ is the polynomial, then $\sum k_r + \sum m_s = n$(order of the linear partial differential equation) and *q* = *r* + *s*. $\alpha_i = \epsilon_i(\theta,\vartheta), \beta_i = \epsilon_i(\theta,\vartheta), \gamma_i = \epsilon_i(\theta,\vartheta)$

Example:

$$\frac{\partial^7 u}{\partial\theta^7} + \frac{\partial^7 u}{\partial\theta^6\partial\vartheta} + 2\frac{\partial^7 u}{\partial\theta^3\partial\vartheta^4} + 2\frac{\partial^7 u}{\partial\theta^2\partial\vartheta^5} - \theta^4\frac{\partial^7 u}{\partial\theta^3\partial\vartheta^4} - \theta^4\frac{\partial^7 u}{\partial\theta^2\partial\vartheta^5} + \frac{\partial^7 u}{\partial\theta^5\partial\vartheta^2}$$

$$+ \frac{\partial^7 u}{\partial\theta^4\partial\vartheta^3} - 4\theta^2\frac{\partial^7 u}{\partial\theta^2\partial\vartheta^5} - 4\theta^2\frac{\partial^7 u}{\partial\theta\partial\theta^6} - \theta^4\frac{\partial^7 u}{\partial\theta\partial\theta^6} - \theta^4\frac{\partial^7 u}{\partial\vartheta^7}$$

$$= DE^3 + D_1Er^3 + 2DE^2F + 2D_1E^2F - \theta^4DEG^2 - \theta^4D_1EG^2 + DE^2G + D_1E^2G$$
$$-4\theta^2DFG^2 - 4\theta^2D_1\boldsymbol{F}G^2 - \theta^4DG^3 - \theta^4D_1G^3 \quad \text{where } \frac{\partial u}{\partial \theta} = D, \frac{\partial u}{\partial \vartheta} = D_1, \frac{\partial^2 u}{\partial \theta^2} = E,$$
$$\frac{\partial^2 u}{\partial \theta \partial \vartheta} = F, \frac{\partial^2 u}{\partial \vartheta^2} = G$$

$$= (E^2 + 2EF + \theta^2EG + EG + 2\theta^2FG + \theta^2G^2)(DE + D_1E - \theta^2DG - \theta^2D_1G)$$

$$= (E + 2F + G)(E + \theta^2G)(E - \theta^2G)(D + D_1) \qquad \ldots\ldots 3.5$$

Order of the linear partial differential equation is 7. The factors of this partial differential equation satisfies the three conditions of the discriminant and so, all three canonical forms with two roots complex conjugates from the irreducible quadratic form, two real unequal roots and two real and equal roots. Thus the **convex region** enclosed by the solution surfaces of this partial differential equation is by the parts of paraboloid, hyperbolid, ellipsoid and a plane.

Outcome: the solution surfaces of an n^{th} order linear partial differential equation encloses a feasible region that is the optimal region satisfies all the boundary conditions, initial conditions that are usually written as the system of linear 1^{st} or 2^{nd} order partial differential equations govern the production function or the activity function.

References:

1. Mat67-Ld-FTA.pdf
2. *B.Fine and G.Rosenberger. The Fundamental Theorem of Algebra. Undergraduate Texts in Mathematics. Springer – Verlag New York*,1997
3. *C. F. Gauss. New Proof of the Theorem That Every Algebraic Rational Integral Function in One Variable can be Resolved into Real Factors of the First or the Second Degree. Ph.D. thesis, University of Helmstedt,* 1799. *Available online at* *http://www.fsc.edu/library/documents/Theorem.pdf*.
4. *The English translation is from Gray. In a footnote, Gray attributes the German quote to "Weber* 1891/92, 19, quoting from a lecture of Kronecker's of 1886". *Gray, Jeremy* (2008), *Plato's Ghost: The Modernist Transformation of Mathematics*, *Princeton University Press*, *p.* 153, *ISBN* 978-1400829040 Weber, Heinrich L. 1891–1892. Kronecker. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 2:5-23. (The quote is on p. 19.)
5. Jump up to:*a b* *Blyth, T.S.; Robertson, E.F. (*2002*). Further Linear Algebra. Springer. pp.* 185–186. *ISBN* 978-1-85233-425-3.
6. *Chang, Sooyoung (*2010*). Academic Genealogy of Mathematicians. World Scientific. pp.* 31–32. *ISBN* 978-981-4282-29-1.
7. *McElroy, Tucker (*2005*). A to Z of mathematicians. Infobase Publishing. pp.* 154–156. *ISBN* 978-0-8160-5338-4.
8. *Gowers, Timothy; Barrow-Green, June; Leader, Imre (*2008*). The Princeton companion to mathematics. Princeton University Press. pp.* 773–774. *ISBN* 978-0-691-11880-2.
9. *Corry, Leo (*2004*). Modern Algebra and the Rise of Mathematical Structures. Birkhäuser. pp. 120.* ISBN 978-3-7643-7002-2.